

# Generalized commutation relations and Non linear momenta theories, a close relationship

Héctor Calisto and Carlos Leiva<sup>‡</sup>

*Departamento de Física, Universidad de Tarapacá, Casilla 7-D Arica, Chile*

February 1, 2008

## Abstract

A revision of generalized commutation relations is performed, besides a description of Non linear momenta realization included in some DSR theories. It is shown that these propositions are closely related, specially we focus on Magueijo Smolin momenta and Kempf et al. and L.N. Chang generalized commutators. Due to this, a new algebra arises with its own features that is also analyzed.

Keywords: DSR, Generalized Commutation Relations, Non commutative.

## 1 Introduction

The standard model is the general framework of our present microscopical theories and it has already been shown that is very successful. Its validity has been confirmed by experiments ranging on a very wide range of energies, from  $eV$  to  $TeV$  [1]. However, from a very recent perspective [2], it seems to be a low energy effective theory, because there are strong difficulties in introducing gravity in the microscopic formulation; on the other hand there are no reasons why it should be the right theory at very high energies and one or all its ingredients (special relativity, quantum mechanics and cluster decomposition principle) could fail at those energies. Indeed, nowadays there are some evidences that Lorentz invariance could not be an exact symmetry at high energies, as recent developments in quantum gravity suggest [3, 4].

Today, one of the principal challenges is to combine successfully the gravitational interactions with Quantum Mechanics. There are direct proposals to do this: M theory and loop quantum gravity are good examples, however we can not say if they are the right choices. Meanwhile the analysis of features that a right theory should have and problems that arise in different efforts, can be very fruitful. A fundamental idea that has been proposed in order to achieve this, is the existence of a fundamental length. A well developed list of arguments

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\*E-mail: hcalisto@uta.cl

†E-mail: cleivas@uta.cl

in order to introduce a minimal length is presented in the Hossenfelder paper [5], a shorter one is the following:

1. String Theory naturally includes a minimal length scale because its very success arises from the fact that interactions do not take place at one point in space time but they occur on the world sheet. Meanwhile, generalized uncertainty relations have aroused and this is not a minor feature.

2. In Loop Quantum Gravity [4] the area operator has a discrete spectrum which gives rise to a smallest-distance structure.

3. Non-Commutative Geometries [6] modify the algebra of the generators of space-time translations such that position measurements fail to commute. The commutator is now proportional to a matrix that has a dimension of  $length^2$

4. A minimal length can be found from the Holographic Principle, which states [7] that the degrees of freedom of a spatial region are determined through the boundary of the region and that the number of degrees of freedom per Planck area is no greater than unity. This leads to a minimal possible uncertainty in length measurements [8].

5. Several phenomenological examinations of possible precision measurements [9], thought experiments about black holes [10] or the general structure of classical [11], semi classical [12] and quantum-foamy space-time [8]. All of them lead to the conclusion that there exists a fundamental limit to distance measurement.

A fundamental minimal length is closely associated with Non commutativity, deformed Heisenberg algebra and DSR theories, all of these theories are also related to Large Extra Dimensions theories and a revision is really fundamental. A brief synopsis of these areas are presented in the next sections 2 and 3, and the relationship between Non linear momenta proposed by Magueijo Smolin and the Generalized Uncertainty Principle is shown in section 4. Finally, a discussion is presented in section 5.

## 2 A brief review on Non commutativity and Deformed commutation relations

The first example of a noncommutative space that was clearly recognized in physics is the quantum phase space. In fact the first considerations on their quantified differential geometry were developed, already 1926, by P.A.M. Dirac [13, 14]. In these works, Dirac discovered the algebraic structure of the quantum phase space postulating its famous rule of quantization of a classic theory, that consists in the replacement of the Poisson brackets of two classical observables by  $i\hbar$  times the commutator of the associated quantum operators. In this way the coordinates of the phase space  $p$  and  $q$  are transformed into noncommutative operators  $\hat{p}$  and  $\hat{q}$  whose commutator is  $i\hbar$ . This Non commutativity implies a relation of incertitude between the operators  $\hat{p}$  and  $\hat{q}$ , that makes disappear the notion of individual points in the phase space being *the Bohr cell* the more plausible idea that survives. In the limit  $\hbar \rightarrow 0$  it recovers the ordinary phase space.

This particular algebra of operators inspired later the more radical idea which consists in the replacement of the coordinates  $x_\mu$  of the space time by noncommutative operators. As it happens in the previous case, the relation  $[x_\mu, x_\nu] \neq 0$  implies an incertitude principle between the coordinates of the space time that makes to disappear the image of point to

short distances. One can argue that as the Bohr cell replaces the points of the classic phase space, the appropriate intuitive notion to replace a point is *the Planck cell* of dimensions given by Planck's area.

Pauli in a revision of the basic principles of quantum mechanics [15] affirmed that only relativistic quantum mechanics is logically complete and expressed vigorously his conviction that new limitations in the possibilities of measurements would have to be expressed more directly in a future theory, and that these would be associated with an essential and deep modification of the basic concepts and the formalism of present quantum theory. Pauli also affirmed that the concepts of space and time on very small scales need a fundamental modification.

The origin of this is the fact that the calculated values of some observables grows to infinity when the continuous theories are extrapolated until arbitrarily small distances, although the measured values are in fact finite. This appeared first in the classical theory of electromagnetism.

Quantum electrodynamics attenuated these divergences, but did not remove them. With the purpose of controlling these divergences, around 1930, Heisenberg [16, 17], proposed to replace the continuous space time by a discrete structure. Nevertheless, at first glance a discrete structure breaks the relativistic invariance that, at that time, was a fundamental requirement of any theory. Later Snyder [18], suggested the idea to use a noncommutative structure, and showed that this necessarily implies the existence of a length scale below which the notion of point does not exist any more. The Snyder method maintains invariance with respect to Lorentz transformations and it is possible that introducing it in a field theory provides an effective cut-off, that is to say, one minimal length in the space time to which the theory is sensible, eliminating therefore the infinities. The Snyder theory later was corrected by Yang [19] to include translational invariance and after some initial variations and developments [20, 21, 22, 23, 24, 25] this idea practically was forgotten, mainly because the renormalization program was revealed appropriate to predict indeed finite numerical values for the observable magnitudes in quantum electrodynamics, without resorting to the non commutativity.

John Von Neumann was the first in trying to describe such quantum spaces rigorously and introduced the term noncommutative geometry to talk about to a geometry in which an algebra of functions is replaced by a noncommutative algebra. The idea of noncommutative geometry were retaken in the Eighties by the French mathematician Alain Connes [26], that generalized the notion of differential structure to the noncommutative case, that is to say, to arbitrary algebras. Connes defined a generalized integration, this took to the description of the noncommutative space time and allowed the definition of theories of field in such spaces.

A different approach has been suggested since long ago, and this idea to generalize Quantum Mechanics via commutation relation has originated from the following arguments:

- Esthetics: There is an obvious, unsatisfying asymmetry between the current treatment of the macro and the micro world, geometry being given a priori at the sub nuclear level. Geometry is assumed to be independent of the physical phenomena.
- Curiosity: Can commutation relations be the same at energies characteristic of atomic at present day physics, that is, over the range of some twelve order of magnitude?.

- Phenomenology: Might the striking high energy phenomena such as quark confinement and the regularities exhibited by the "heavy photons" be connected with the geometry of space?.

Then to do research on the above mentioned questions [30] the following generalization of the commutation relation was proposed:

$$[q, p] = i\hbar + \frac{i\ell}{c}\mathcal{F}(q, p) \quad (2.1)$$

where  $\ell$  is a constant with dimensions of length which is sufficiently small so that the second term in the right hand side is negligible except for very high energy or momentum, in other words, for low energies the usual Heisenberg commutation relation holds, and  $c$  is the velocity of light in vacuum.

We assume that the variable  $p$  well understood "the momentum measured in the laboratory" and therefore we consider equation (1) as an equation which determines the position operator  $q$  when  $\mathcal{F}(q, p)$  is given. In the low energy or low momentum limit  $q = i\hbar d/dp$  is the usual position operator.

The function  $\mathcal{F}(q, p)$  should in general depend on the dynamics of the problem, and therefore the operator  $q$  and the  $q$ -eigenvalues, this is to say, the physical space, is not given a priori but is determined by the physics of the problem as represented by the choice of  $\mathcal{F}(q, p)$ . Hence the name to describe this procedure: Dynamical Quantization.

We further assume the existence of a Hamiltonian function  $H = H(q, p)$ , and the validity of the Heisenberg equation of motion:

$$\frac{d\Omega}{dt} = \frac{i}{\hbar} [H, \Omega] \quad (2.2)$$

where  $\Omega$  is a dynamical variable and  $H$  is the Hamiltonian of the system, which continues being valid.

Next we will briefly survey some previous results, with some of the possible elections of  $\mathcal{F}(q, p)$  both, for the non-relativistic case and a possible relativistic generalization. The simplest choice for  $\mathcal{F}(q, p)$  is  $H(q, p)$  which leads to the commutation relation:

$$[q, p] = i\hbar + \frac{i\ell}{c}H(q, p) \quad (2.3)$$

As was postulated in [27, 28]. Equation (2.3) implies

$$\Delta p \Delta q \geq \left| \frac{\hbar}{2} + \frac{\ell}{2c}E \right| \quad (2.4)$$

then the lower bound of  $\Delta p \Delta q$  grows linearly with energy, except for logarithmical corrections as it was confirmed by Giffon and Predazzi [31] using high energy data, which preliminarily yielded  $\ell = 2.3 \times 10^{-6}\text{fm}$ . The harmonic oscillator problem can still be solved exactly with equation (2.3), yielding a non-equal spacing law for the energy levels [28].

$$E_n = \frac{\hbar\omega}{2\alpha} \left[ \frac{2 - \left(\frac{\beta}{\alpha}\right)^n - \left(\frac{\beta}{\alpha}\right)^{n+1}}{1 - \frac{\beta}{\alpha}} \right] \quad (2.5)$$

where  $\alpha = 1 - \omega\ell/2c$  and  $\beta = 1 + \omega\ell/2c$ .

In the one dimensional case, for a non relativistic free particle we have

$$[q, p] = i\hbar(1 + \delta^2 p^2) \quad (2.6)$$

where  $\delta^2 = \ell/2m\hbar c$ . The corresponding position operator eigenvalues problem leads to the physical space. The result was that the spectrum is discrete [29]

$$\lambda_n = 2n\hbar\delta, \quad n = 0, \pm 1, \pm 2, \dots \quad (2.7)$$

That is to say, the space generated by  $H = p^2/2m$ , is a one-dimensional lattice, the smallest interval being

$$\Delta q_{min} = 2\hbar\delta = \sqrt{\frac{2\hbar\ell}{mc}} \quad (2.8)$$

Then, once the particle has been located at any arbitrary point of the one dimensional space, the rest of the space "feels" it, the lattice appears instantaneously; in this sense, the geometry acts as constant force, a linear potential. Further, if  $\Delta q_{min}$  is the spatial extension of an extended object, then it is meaningless to ask for its "constituents", that is, objects of smaller size, because no test particle can go into it.

For a relativistic generalization our starting point is the observation that the function  $\mathcal{F}$  takes place in (6) and can be written as follows

$$\mathcal{F} = \frac{p^2}{2m} = \frac{1}{2m} p \cdot p \quad (2.9)$$

which suggests to choose, for the relativistic free particle case

$$\mathcal{F} = \frac{p_\mu p_\nu}{2m} \quad (2.10)$$

Equations (1) and (10) then lead to the following generalization

$$[q_\mu, p_\nu] = -i\hbar (g_{\mu\nu} - \delta^2 p_\mu p_\nu) \quad (2.11)$$

where  $g_{00} = -g_{kk} = 1$  for  $k = 1, 2, 3$  and  $g_{\mu\nu} = 0$  para  $\mu \neq \nu$ . The implication of this commutation relation can be investigated. Assuming as before that the  $p_\mu$  are given we find that

$$q_\mu = -i\hbar (g_{\mu\nu} - \delta^2 p_\mu p_\nu) \frac{\partial}{\partial p_\nu} + i\hbar \kappa p_\mu \quad (2.12)$$

where  $\kappa$  is a real positive constant which determines the weight function in the inner product if we wish to give a physical meaning to our operators. By imposing  $(q_\mu \phi, \psi) = (\phi, q_\mu \psi)$  we find that the suitable internal product is

$$(\psi, \phi) = \int d\tau \frac{\psi^* \phi}{(1 - \delta^2 g_{\mu\nu} p_\mu p_\nu)^{1-\gamma}} \quad (2.13)$$

where  $d\tau$  is an appropriate volume element,  $\gamma = \frac{\kappa}{\delta^2} - \frac{1}{2}(D-1)$  and  $D$  is the dimension of space time. We remark that since  $\kappa$  is a free parameter we can choose  $\beta$  so that the internal product normally used in quantum mechanics continues being valid.

The main results of this generalization are:

- Coordinates operators do not commute, their commutators being proportional to the infinitesimal generators of the Lorentz group.
- The spatial operators  $q_1, q_2, q_3$  having discrete spectrum.
- The time operator  $q_0$  has a continuum spectrum.

Applications of this type of generalization, show that the energy spectrum of the one dimensional harmonic oscillator is given by

$$E_n = \hbar\omega \left[ \left( n + \frac{1}{2} \right) \sqrt{1 + \left( \frac{\omega\ell}{2c} \right)^2} \right] + \hbar\omega \left[ \left( n^2 + n + \frac{1}{2} \right) \frac{\omega\ell}{2c} \right] \quad (2.14)$$

and the eigenfunctions are the Gegenbauer Polynomials. In these calculations we have used the one dimensional version of the position operator 2.12.

The eigenvalues problem for the three-dimensional harmonic isotropic oscillator can be solved using the three-dimensional version of position operator 2.12. We found

$$E_n = \hbar\omega \left( n + \frac{3}{2} \right) \sqrt{1 + \left( \frac{\omega\ell}{2c} \right)^2} + \hbar\omega \left( n^2 + 3n - s(s+1) + \frac{3}{2} \right) \frac{\omega\ell}{2c} \quad (2.15)$$

where  $s(s+1)$  are the usual eigenvalues for the angular orbital momentum of the oscillator. The eigenfunctions are the Jacobi Polynomials.

Following suggestions originated in quantum theory of gravity and string theory, Kempf, Mangano and Mann [31] have suggested a generalized uncertainty relation of the form

$$[\mathbf{x}, \mathbf{p}] = -i\hbar \left( \mathbf{1} + \alpha \mathbf{x}^2 + \beta \mathbf{p}^2 \right) \quad (2.16)$$

where  $\alpha, \beta$  are positive and independent parameters. They discussed some consequences of this generalization in non relativistic quantum mechanics and have worked in detail the case  $\alpha = 0$ . Their result are very similar to those found in [20]. Following this same type of suggestions and with similar results, Chang and coworkers also have developed some applications to non relativistic quantum mechanics [32, 33] and they have discussed some consequences of the non commutativity in classical mechanics [34]. Preliminary applications to relativistic quantum mechanics and quantum fields theory have been developed recently by Yamaguchi in a Riemannian energy-momentum space [35, 36] and by Ahluwalia-Khalilova[37].

The non commutative theories also play an important role in the area of condensed matter, which is not only the concrete accomplishment of the mathematical models used to explore the properties of space time in physics at high energies and quantum theory of gravity, but represents concrete applications in an area of increasing interest and impact. A classical example is the theory of an electron in an external magnetic field, projected on the lower Landau level, which can be treated like a non commutative theory. It is by this reason that these ideas are relevant for the study of the quantum Hall effect [38] and in fact they have been very useful in this context [39].

A recent and very convincing example of a noncommutative theory in the area of condensed matter is the quantum theory of mesoscopic electrical circuits developed by Li and

Chen [40, 41] that takes into account the discretization of the electric charge which leads to a new commutation relation between the charge and current operators, that is similar to the one studied in physics at high energies and quantum theory of gravity. Several advances and applications in the context of the mesoscopic circuits with discrete charge have been made by J.C. Flores [42] and by J.C. Flores in collaboration with C. Utreras [43].

Summarizing, noncommutative theories have been revealed as tools of certain utility in theoretical physics. They appear as much in the physics of high energies, for the description from a fundamental level of the space time on small scale, as also in the area of condensed matter to describe the Hall effect and in the quantum theory of the mesoscopic electrical circuits. The enormous activity around these theories mainly is bound to the appearance of non commutativity in the limit of low energies of string theory mentioned previously. Since string theory is the only well-known theory that it could unify all the fundamental interactions, it is possible that the problems of control of divergences in the quantum theory of field and quantization of gravity are basically related by means of some sort of non commutative.

### 3 DSR theories

Absolute values of length, time or energy are not, at first glance, in agreement with the Lorentz transformations, this point has produced the idea of modifying Lorentz symmetry.

Several solutions to the problem on how to modify the Lorentz boosts, have been proposed. In particular, a very interesting solution was given by Doubly (or Deformed), Special relativity (DSR) theories [44, 45, 46]. These theories are based on a generalization of Lorentz transformations through the more broad point of view of conformal transformations, they have two observed independent scales: velocity of light and Planck length.

Usually a modification of Lorentz boosts in momentum space is performed, however when this is done, retrieving the position space dynamics can be a very hard task, due to the loss of linearity. Kimberly, Magueijo and Medeiros [47], have proposed some methods to undertake this problem by using a free field theory. So, this is a worth research aspect of DSR theories, that is not well understood until today.

Another approach can be seen in [48], on the other hand, the approach of Deriglazov [49] is very interesting because he starts from a conformal group, but the idea is different from the one proposed here, because it is based in position space and the problem of retrieving the position space dynamics is not present in that work.

Some approaches have been performed in order to identify AdS spaces as arenas for DSR theories and the approach of this paper could shed some light on that problem too.

Even though DSR theories were of increasing interest because they could be useful as a new tools in gravity theories, in Cosmology as an alternative to inflation [50, 51], and in other fields like propagation of light [52], that is related, for instance, to cosmic microwave background radiation, nowadays they fell out of favor with researches because there are serious conceptual issues that DSR has so far failed to address like:

- They seem to belong to a trivial  $k$  deformations of Poincaré algebra [53]
- Special relativity extensions do not need to have a non linear character. [54]

- They have very hard problems with multi particle sector (the well known soccer ball problem) [55]
- A fundamental length scale does not violate Lorentz invariance as Snyder himself shown first [18] and can be seen also in [56]
- Lorentz violations due to the combination of some consequences of DSR like theories and elementary particle interactions are at the percent level some 20 orders of magnitude higher than expected unless a very unnatural tuning is performed [57]

Despite all these problems, we will show that a conspicuous feature of these theories is closely related to Generalized Commutation Relations, the Magueijo Smolin non linear momenta that are still fashionable.

About this feature, Leiva [58, 59, 60] has shown that it is possible from a formal point of view to obtain Fock-Lorentz [61, 62] and Magueijo-Smolín deformations, through a reduction process and using conformal group generators as the generators of the deformed Lorentz algebra. Then, the open problem of obtaining the space time dynamics is solved through the relationship, on the physical surface, between momenta and velocities that rise up from this method.

More specifically, it was conjectured that the deformations of the Lorentz algebra performed in the position space (the Fock-Lorentz formulation), can be treated as a transformation made by a linear combination of conformal group generators. On the other hand, the momentum case (the Magueijo Smolin formulation), can be understood as an analog process, but the inclusion of a new generator is needed. This new generator can be obtained from the same theory, and it completes the set of symmetries of the massless Klein Gordon equation. Then, a DSR massless particle is shown to be isomorphic to a normal Lorentz particle living in a  $d + 2$  space, and the deformations are induced by dimensional reduction.

## 4 DSR and Generalized Commutation Relations

It is possible to show a relationship between non commutativity, deformed algebra and DSR theories. We are going to show that a deformed algebra in the sense of [44, 45] and others, can be seen as a first order approximation to some DSR theories. In fact, if the commutator between the position operator and the Magueijo-Smolín momentum operator  $\pi$  is calculated, we obtain:

$$\begin{aligned}
 [x_i, p_j] &= i\delta_{ij}, \quad i, j = 1, \dots, n \\
 \pi_j &= \frac{p_j}{(1 - l_p p^2)}, \\
 [x_i, \pi_j] &= i \frac{\delta_{ij}}{(1 - l_p p^2)} + i \frac{2l_p p_i p_j}{(1 - l_p p^2)^2}.
 \end{aligned} \tag{4.1}$$

and using the definition of  $\pi_j$ :

$$[\pi_i, \pi_j] = 0 \tag{4.2}$$



$$[x_i, \pi_j] = 2i\delta_{ij}l_p f + 2il_p\pi_i\pi_j \quad (4.3)$$

where

$$f = \frac{\pi^2}{\sqrt{1 + 4l_p\pi^2} - 1} \quad (4.4)$$

at first order gives:

$$[x_i, \pi_j] \approx i\delta_{ij}(1 + \ell_p\pi^2) + 2il_p\pi_i\pi_j \quad (4.5)$$

and that exactly is the relation proposed by Kempf et al. and L.N. Chang.

On the other hand, using a very similar treatment to the one performed by Leiva to obtain the DSR generators, J. Romero and A. Zamora [63] obtained the Snyder commutation relations. This is a new clue in that in all these theories there is a very important underlying relationship.

The full position operator that satisfies 4.1, in the  $\pi$ -representation, can be easily calculated and we find

$$x_i = 2i\ell_p f \frac{\partial}{\partial \pi_i} + 2i\ell_p \pi_i \left( \pi_j \frac{\partial}{\partial \pi_j} \right) + 2i\ell_p \kappa \pi_i \quad (4.6)$$

where  $\kappa$  is a free parameter that can be fixed in the definition of the internal product since as it can be easily verified the operators  $x_i$  given in (21) are not hermitian with the internal product usually employed in quantum mechanics:

$$(\psi, \phi) = \int d\pi_1 \dots d\pi_n \psi^* \phi \quad (4.7)$$

consequently, we needed to build an internal product in which the operators  $x_i$  are self adjoint. We postulate the general form:

$$(\psi, \phi) = \int d\pi_1 \dots d\pi_n \frac{\psi^* \phi}{W(\pi \cdot \pi)} \quad (4.8)$$

where  $W(\pi \cdot \pi)$  is a weight function to be determined and  $\pi \cdot \pi = \pi_i \pi_i$ . Imposing the condition of hermiticity of the operator  $x_i$  with this new internal product:

$$(x_i \psi, \phi) = (\psi, x_i \phi) \quad (4.9)$$

and requiring that the functions  $\psi$  and  $\phi$  vanish suitably in the infinite, after an integration by parts, we find that 4.9 is fulfilled, if the weight function  $W$  satisfies the following differential equation:

$$\begin{aligned} & f \frac{\partial W}{\partial \pi_i} + \pi_i \left( \pi_j \frac{\partial W}{\partial \pi_j} \right) \\ & + \left( -\frac{\partial f}{\partial \pi_i} + 2\kappa\pi_i - (n+1)\pi_i \right) W = 0 \end{aligned} \quad (4.10)$$

At this point and to simplify the equation for  $W$  we choose  $\kappa = (n+1)/2$ . Now using the function  $f$  gives in 4.4 we can see that:

$$\frac{\partial f}{\partial \pi_i} = \frac{2\pi_i (1 + 2\ell_p\pi^2 - \sqrt{1 + 4\ell_p\pi^2})}{(\sqrt{1 + 4\ell_p\pi^2} - 1)^2 \sqrt{1 + 4\ell_p\pi^2}} = 2\pi_i \frac{\partial f}{\partial \pi} \quad (4.11)$$

Then it is evident that we can choose  $W$  as a function of  $\pi = \sqrt{\pi_1^2 + \dots + \pi_n^2}$ . With this  $\partial W / \partial \pi_i = (\pi_i / \pi) \partial W / \partial \pi$  and the equation for  $W$  in spherical coordinates becomes

$$\left( \frac{\sqrt{1 + 4\ell_p \pi^2}}{\sqrt{1 + 4\ell_p \pi^2} - 1} \right) \pi \frac{dW}{d\pi} - \frac{2(1 + 2\ell_p \pi^2 - \sqrt{1 + 4\ell_p \pi^2})}{(\sqrt{1 + 4\ell_p \pi^2} - 1)^2 \sqrt{1 + 4\ell_p \pi^2}} W = 0 \quad (4.12)$$

this last equation can be integrated directly giving

$$W(\pi) = \frac{\sqrt{1 + 4\ell_p \pi^2}}{1 + \sqrt{1 + 4\ell_p \pi^2}} \quad (4.13)$$

where a multiplicative integration constant has been chosen equal to one because any constant of this type always can be incorporated in the normalization constants of the wave functions  $\psi, \phi$ .

In order to complete this section we define the angular momentum operators in the usual way as:

$$L_j = \epsilon_{j k \ell} x_k \pi_\ell \quad (4.14)$$

where  $\epsilon_{j k \ell}$  is the usual Levi-Civita symbol. Doing simple algebraic manipulations we find that  $L_j$  is explicitly given by:

$$L_j = -2i \ell_p f \epsilon_{j k \ell} \pi_k \frac{\partial}{\partial \pi_\ell} \quad (4.15)$$

In addition, other direct calculations allow to show that:

$$[x_j, x_k] = 2i \ell_p \left[ 1 + 2 \left( 1 + \frac{\pi^2}{f} \right) \frac{\partial f}{\partial \pi} \right] \epsilon_{j k \ell} L_\ell \quad (4.16)$$

The position operators do not commute. Evidently:

$$\Delta x_j \Delta x_k \neq 0 \quad (4.17)$$

as has been usual in this type of theories.

In this way equations (4.2),(4.3),(4.16) are the commutators of a new algebra that comprises Non linear momenta in the way proposed by Magueijo Smolin and GUP theories.

## 5 Discussion and outlook

In section 2 we have reviewed the principal arguments and ways to introduce extra terms in the canonical relations. In section 3, we have seen some facts of DSR theories and in section 4 we have seen that how Magueijo Smolin momenta produce non linear commutation relations that are, to first order, the same of those proposed by Kempf et al. and Chang, used also by Ahluwalia-Khalilova in an relativistic extension. Furthermore we constructed explicitly the position operator and investigated its features.

We think it is very important to study features that arise when a minimal length or a Generalized Uncertainty Principle or Non commutativity are introduced in areas like Classical Mechanics and Quantum Mechanics, because new hidden or exotic symmetries could emerge. [64]

Finally, to extend the study to quantum field formulation could be also a interesting task, indeed Lorentz violation in supersymmetric Field Theories was investigated by Nibbelink and Pospelov [65]. The extension of these theories to Quantum Field Theory has been done just for the noncommutative case [66, 67], but they have not been deeply investigated, there is just an extension proposed by Magueijo [68].

### Acknowledgements

It is a pleasure for us to thank Mauro Bologna and Fernando Mora by its valuable suggestions and critics. Also we thank to the Research Grant 4723-05 from UTA which partially supported this research.

## References

- [1] J.M. Carmona and J.L. Cortés, *Infrared and ultraviolet cutoffs of quantum field theory*, Phys. Rev. D 65 (2002) 025006 [arXiv:hep-th/0012028]
- [2] S. Weinberg. *The quantum field theory of fields*, Cambridge University Press, Cambridge 199.
- [3] G. Amelino-Camelia, J. Ellis, N.E. Mavrountzos, D.V. Naonopoulos and S. Sarkar, *Test of quantum gravity from observations of  $\gamma$ -ray bursts*, Nature 393 (1998) 763.
- [4] C. Rovelli, Living Rev. Rel. 1,1 (1998); A. Perez, Class. Quant. Grav. 20. R43 (2003); T. Thiemann, Lect. Notes Phys. 631, 41 (2003); A Ashtekar and J. Lewandowski, Class. Quant. Grav. 21, R53 (2004).
- [5] Hossenfelder, Sabine [arXiv:hep-ph/0410122]
- [6] R.J. Szabo, Phys. Rept. 378, 207 (2003).
- [7] G. 'tHooft, [arXiv:gr-qc/9310026], L. Susskind, J. Math. Phys. 36, 6377 (1995).
- [8] Y. J. Ng and H. van Dam, [arXiv:gr-qc/0403057].
- [9] E. P. Wigner, Rev. Mod. Phys. 29,255 (1957), H. Salecker and E. P. Wigner, Phys. Rev. 109,571 (1958), J. D. Barrow, Phys. Rev. D 54, 6563 (1996), N. Sasakura, Prog. Theor. Phys. 102, 169 (1999).
- [10] F. Scardigli, Phys. Lett. B 452, 39 (1999); T. Padmanabhan, T. R. Seshadri and T. P. Singh, Int. J. Mod. Phys. A 1, 491 (1986).

- [11] X. Calmet, M. Graesser and S. D. H. Hsu, [arXiv:hep-th/0405033]; V. Daftardar and N. Dadhich, Int. J. Mod. Phys. A 1, 731 (1986); Y. J. Ng and H. van Dam, Mod Phys. Lett. A 9, 335 (1994).
- [12] -C. I. Kuo and L. H. Ford, Phys. Rev. D 47, 4510 (1993).
- [13] P. A. M. Dirac, *The fundamental equations of quantum mechanics*, Proc. Roy. Soc. **A109**, 642 (1926).
- [14] P. A. M. Dirac, *On quantum algebras*, Proc. Camb. Phil. Soc. **23**, 412 (1926).
- [15] W. Pauli: *Die allgemeinen Prinzipien der Wellenmechanik* in Handb.d.Physik (Springer, Heidelberg, 1933), 24/1, p. 246, 247, 271, 272.
- [16] W. Heisenberg, *Die Leobachtbaren Grossen in der Theorie der Elemntarteilchen*, Z Phys **120**, 513 (1943)
- [17] W. Heisenberg, *Quantum Theory of Fields and Elementary Particles*, Rev Mod Phys,**29**, 269 (1957)
- [18] H. S. Snyder, *Quantized Space-Time*, Phys. Rev. **71**, 38 (1947).
- [19] C. N. Yang, *On Quantized Space-Time*, Phys. Rev. **72**, 874 (1947).
- [20] H. S. Snyder, *The Electromagnetic Field in Space-Time*, Phys. Rev. **72**, 68 (1947).
- [21] A. Schild, *Discrete Space-Time and Integral Lorentz Transformations*, Phys. Rev. 73, 414-415 (1948). The Quantization of Space and Time H. T. Flint Phys. Rev. 74, 209-210 (1948)
- [22] R. J. Finkelstein, *On the Quantization of a Unitary Field Theory*, Phys. Rev. 75, 1079-1087 (1949)
- [23] E. J. Hellund and K. Tanaka, *Quantized Space-Time*, Phys. Rev. 94, 192-195 (1954)
- [24] E. L. Hill, *Relativistic Theory of Discrete Momentum Space and Discrete Space-Time*, Phys. Rev. 100, 1780-1783 (1955)
- [25] M. R. Hamilton and G. Sandri, *Class of Fields in Snyder Spaces*, Phys. Rev. D 8, 1788-1795 (1973)
- [26] A. Connes, *Gravity coupled with matter and the foundation of non-commutative geometry*, (1996); hep-th/9603053.
- [27] I. Saavedra, *A generalization of Quantum Mechanics for High Energies and Quarks Physics*, in Quantum Theory and the Structures of Time and Space, Vol. IV, Edited by L. Castell, M. Drieschner, and C. von Weizsacker, Max Planck Intitut, Carl Hauser Verlag, Munich (1981).
- [28] I. Saavedra and C. Utreras, *A generalization of Quantum Mechanics for High Energies and Quarks Physics*, Phys. Lett., 98 B, 74 (1981).

- [29] S. Montecinos, I. Saavedra and O. Kunstmann, *High Energy Quantum Mechanics and Dynamical Quantization*, Phys. Lett. 109 A, 139 (1985).
- [30] M. Giffon and E. Predazzi, *Quantum mechanics and high energy physics*, Lett. Nuovo Cimento **37**, 430 (1983).
- [31] A. Kempf, G. Mangano, and R. B. Mann, *Hilbert space representation of the minimal length uncertainty relation*, Phys. Rev. D 52, 1108-1118 (1995)
- [32] L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, *Exact solution of the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations*, Phys. Rev. D 66, 026003 (2002).
- [33] S. Benczik, L. N. Chang, D. Minic and T. Takeuchi, *The hydrogen atom with minimal length*, arXiv:hep-th/0502222.
- [34] S. Benczik, L. N. Chang, D. Minic, N. Okamura, S. Rayyan, T. Takeuchi, *Classical implications of the minimal length uncertainty relation*, arXiv:hep-th/0209119.
- [35] Y. Yamaguchi, *Quantum mechanics and quantum field theories in the Quantized Space. I*, Prog. Theor. Physics, Vol.111 No.4, 545 (2004).
- [36] Y. Yamaguchi, *Quantum mechanics and quantum field theories in the Quantized Space. II*, Prog. Theor. Physics, Vol.111 No.4, 545 (2005).
- [37] D.V. Ahluwalia, Phys Lett. A 275 (2000) 31-35
- [38] S. Girvin and R. Prange, *The quantum Hall effect*, Springer New York (1987).
- [39] J. Bellisard, A. van Elst and H. Schultz-Valdes, *The noncommutative geometry and the quantum Hall effect*, (1993); con-mat/93011005.
- [40] Y. Q. Li and B. Chen, *Quantum theory for mesoscopic electric circuits*, Phys. Rev. **B 53**, 4027 (1996).
- [41] Y. Q. Li, *Commutation Relations in Mesoscopic Electric Circuits*, 2000; con-mat/0009352.
- [42] J. C. Flores, *Mesoscopic circuits with charge discreteness: Quantum transmission lines*, Phys. Rev. **B 64**, 235309 (2001).
- [43] J. C. Flores and C. A. Utreras, *Mesoscopic circuits with charge discreteness: Quantum current magnification for mutual inductances*, Phys. Rev. **B 66**, 153410 (2002).
- [44] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403(2002) [arXiv:hep-th/0112090]
- [45] G. Amelino-Camelia, Int. J. Mod. Phys. D11, 35, 2002, [arXiv:gr-qc/0012051], Ohys. Lett. B510, 255-263, 2001.
- [46] J. Kowalski-Glikman, Phys. Lett. A286, 391-394, 2001, [arXiv:hep-th/0102098]; N.R. Bruno, G. Amelino-Camelia, J. Kowalski-Glikman, Phys. Lett. B522, 133-138, 2001.

- [47] D. Kimberly, J. Magueijo and J. Medeiros [arXiv:gr-qc/0303067]
- [48] Sijie Gao and Xiaoning Wu, *Position space of Doubly Special Relativity* [arXiv:gr-qc/0311009]
- [49] A. A. Deriglazov, Doubly special relativity in position space starting from the conformal group, hep-th/0409232.
- [50] J. Moffat, Int. J. of Physics D **2**, 351 (1993); J. Moffat, Foundations of Physics, **23**, 411 (1993)
- [51] A. Albrecht and J. Magueijo, Phys. Rev. D **59**, 043516 (1999)
- [52] Sung Ku Kim, Sun Myong Kim, Chaiho Rim and Jae Hyung Yee, *Propagation of Light in Doubly Special Relativity* [arXiv:gr-qc/0401078]
- [53] D. Grumiller, W. Kummer, D.V. Vassilevich, Ukr. J. Phys. 48 (2003) 329-333;[arXiv:hep-th/0301061].
- [54] D. V. Ahluwalia-Khalilova, Class. Quant. Grav. 22 (2005) 1433-1450;[arXiv:hep-th/050314].
- [55] R. Schutzhold, W.G. Unruh, JETP Lett. 78: 431-435, 2003, Pisma Zh. Eksp. Teor. Fiz. 78: 899-903, 2003.
- [56] C. Chryssomalakos, E. Okon, Int. J. Mod. Phys. D13: 2003-2034, 2004 ;[arXiv:hep-th/0410212].
- [57] John Collins et al., Phys Rev. Lett. 93: 191301, 2004;[arXiv:gr-qc/0403053].
- [58] C. Leiva and M. S. Plyushchay, *Conformal Symmetry of Relativistic and Nonrelativistic Systems and AdS/CFT Correspondence*, Annals of Physics 307 0310 (2003) 372-391. [arXiv:hep-th/0301244].
- [59] C. Leiva, *Conformal Generators and Doubly Special Relativity Theories*, Mod. Phys. Lett. A, Vol. 20, No. 11 (2005) pp. 861-867, [arXiv: gr-qg/0405149].
- [60] C. Leiva (Extended Conformal Group and DSR Velocities on the Physical Surface , [arXiv: hep-th/0410159].
- [61] V. Fock, *The theory of space-time and gravitation*, Pergamon Press,1964.
- [62] S.N. Manida, [arXiv:gr-qc/9905046]
- [63] Juan M. Romero and Adolfo Zamora, *Snyder noncommutative space-time from two-time physics*, [arXiv:hep-th/0408193]
- [64] P.A. Horvathy, Exotic Galilean symmetry and non-commutative mechanics in mathematical and in condensed matter physics.[arXiv:hep-th/0602133 ]

- [65] Stefan Groot Nibbelink and Maxim Pospelov *Lorentz Violation in Supersymmetric Field Theories*, [arXiv:hep-ph/0404271].
- [66] J. Gamboa, M. Loewe and F. Méndez, *Relativistic Invariance, Multiverse and Quantum Field Theory*, [aXiv:hep-th/0311014]
- [67] J. M. Carmona, J.L. Cortés, J. Gamboa and F. Méndez, *Quantum theory of noncommutative fields*, JHEP 0303:058,2003 [arXiv: hep-th/0301248]
- [68] D. Kimberly, J. Magueijo, J. Medeiros, *Non-Linear relativity in Position Space*, Phys. Rev. D 70 (2004) 084007, [arXiv:gr-qc/0303067]